Problem Set 3

Macroeconomics III

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Problem 1

The economy

- Population growth: $n > 0$
- Capital depreciation: $1 > \delta > 0$
- Tax on capital income: τ
- Real after tax interest income become: $(1-\tau)r_t$
- Lump-sum transfer: T_t such that $T_t = \tau r_t k_t$

Households maximize discounted utility:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t (1+n)^t u(c_t) \tag{1}$$

The dynamic budget constraint becomes:

$$(1+n)k_{t+1} = (1+(1-\tau)r_t - \delta)k_t + w_t + T_t - c_t \tag{2}$$

Problem 1

We assume that markets are perfectly competitive such that:

$$w_t = \frac{\partial F(K_t, L_t)}{\partial L_t} = f(k_t) - f'(k_t)k_t$$
$$r_t = \frac{\partial F(K_t, L_t)}{\partial K_t} = f'(k_t)$$

Note that the following holds (and implies zero profits):

$$r_t k_t + w_t = f'(k_t)k_t + f(k_t) - f'(k_t)k_t = f(k_t) \tag{3}$$

The households take T_t as exogenously given and do not take into account that the taxes they pay are given back to them by lump-sum transfers. **Why? What is the intuition?**

The capital income tax from one individual doesn't affect the government budget as it is too small. Furthermore, the taxes are paid back in lump-sum transfers, which doesn't depend on the amount of taxes paid by the individual. Hence, the capital income tax paid by individual i does not affect the size of the transfer that individual i receives.

Problem 1a - Lagrangian and FOCs

Write the Lagrangian for the households' optimization problem, find the first order conditions that characterize the behavior of households, and from these the Euler equation (the Keynes Ramsey rule). How does the tax rate affect it?

The Lagrangian is given by:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t (1+n)^t \Big[u(c_t) + \lambda_t \Big(\big(1-\delta+(1-\tau) r_t\big) k_t + w_t + T_t - (1+n) k_{t+1} - c_t \Big) \Big]$$

The first order conditions are:

$$\text{FOC wrt. } c_t: \qquad u'(c_t) = \lambda_t$$

$$\text{FOC wrt. } k_{t+1}: \qquad \lambda_t = \beta \lambda_{t+1} \big[1 - \delta + (1-\tau) r_{t+1} \big]$$

Problem 1a - Euler equation and effect from taxes

Write the Lagrangian for the households' optimization problem, find the first order conditions that characterize the behavior of households, and **from these the Euler equation (the Keynes Ramsey rule). How does the tax rate affect it?**

Combining the two leads to the Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})\left[1 - \delta + (1-\tau)r_{t+1}\right]$$

The Euler equation dictates that the optimal consumption path of the households balances the marginal utility today with the marginal utility from the next period that could be achieved by postponing consumption.

The tax on capital income lowers the interest received by the household, why it becomes less attractive to save money. Therefore, a positive capital income tax leads to higher consumption today.

Problem 1b - C-locus

How does capital taxation affect the curves that characterize steady state in the phase diagram?

The c-locus is determined by the Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})\big[1 - \delta + (1-\tau)r_{t+1}\big]$$

Imposing constant consumption, $c_t = c_{t+1}$, leads to:

$$\begin{aligned} 1 &= \beta\big[1 - \delta + (1-\tau)r_{t+1}\big] \\ 1 &= \beta\big[1 - \delta + (1-\tau)f'(k_{t+1})\big] \end{aligned} \tag{4}$$

The capital level that ensures that c is in equilibrium is such that:

$$f'(k^*) = \frac{1 - \beta(1-\delta)}{\beta(1-\tau)}$$

Hence, an increase in the tax level increases the marginal product of capital in steady state. Since the production function is concave, the steady state level of capital decreases as capital income tax increases.[1]

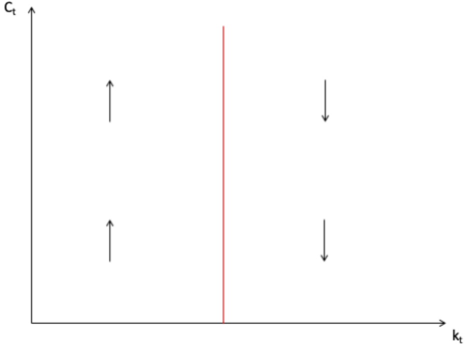
[1] We assume concavity of $f(k)$.

Problem 1b - C-locus dynamics

We investigate the dynamics of c_t as in problem set 2. Suppose c_t grows, $c_{t+1} > c_t$, such that $\frac{u'(c_t)}{u'(c_{t+1})} > 1$. Then it follows from the Euler equation:

$$\beta[1-\delta+(1-\tau)f'(k_{t+1})] > \beta[1-\delta+(1-\tau)f'(k^*)]$$
$$f'(k_{t+1}) > f'(k^*) \quad \xRightarrow[\text{concave } f]{} \quad k_{t+1} < k^*$$

Hence, consumption is increasing when $k_{t+1} < k^*$ and vice versa.



Problem 1b - K-locus

The K-locus is determined by the budget constraint:

$$(1+n)k_{t+1} = (1+(1-\tau)r_t - \delta)k_t + w_t + T_t - c_t$$

Rewriting yields:

$$\begin{aligned}
(1+n)k_{t+1} &= (1+(1-\tau)r_t - \delta)k_t + w_t + T_t - c_t \\
c_t &= (1+(1-\tau)r_t - \delta)k_t + w_t + T_t - (1+n)k_{t+1} \\
c_t &= (1-\delta)k_t + \underbrace{r_t k_t + w_t}_{=f(k_t)} - \tau r_t k_t + \underbrace{T_t}_{=\tau r_t k_t} - (1+n)k_{t+1} \\
c_t &= f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}
\end{aligned}$$

We then assume constant capital, $k_t = k_{t+1} = k$:

$$c = f(k) - (n+\delta)k \tag{5}$$

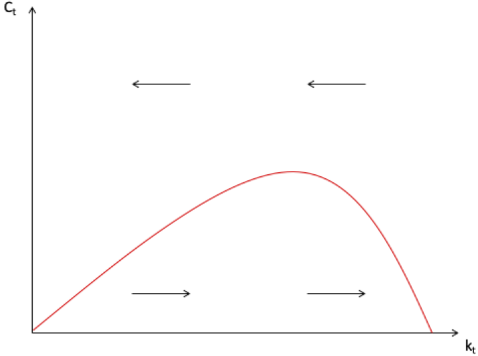
The k-locus does not depend on taxes since all the tax revenue is rebated and capital accumulation is, therefore, unaffected.

Problem 1b - K-locus dynamics

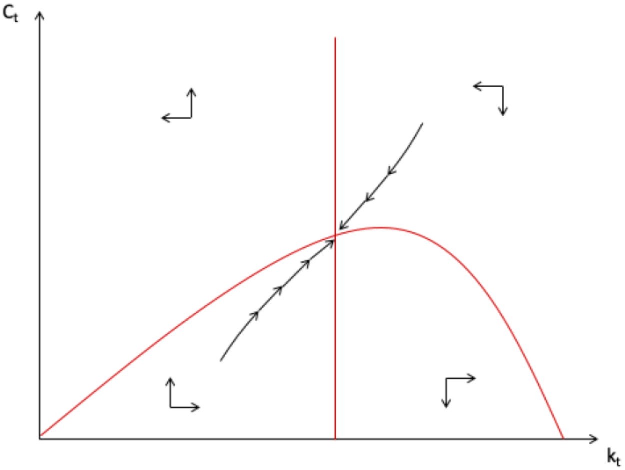
We investigate the dynamics of k_t as in problem set 2. Suppose we're in a situation where $k_{t+1} > k_t$:

$$c_t = f(k_t) + (1 - \delta)k_t - (1 + n)k_{t+1} < f(k^*) - (\delta + n)k^*$$

k_{t+1} only enters negatively in the expression for c_t, why the $<$ holds. Hence, $k_{t+1} > k_t$ when c_t is below the k-locus such that:



Problem 1b - Phase diagram



Note: The line with arrows is called the saddle path.

The economy will converge towards (k^*, c^*) along the saddle path. If the economy is not on the saddle path, the economy will diverge.

Problem 1c - Introducing capital income tax

Assume that initially the economy is in steady state with no capital taxation. How does the economy react to the unexpected introduction of the tax? How does capital and consumption respond to the introduction of the tax, and how does the new steady state compare to the situation without taxation?

$$1 = \beta\big[1 - \delta + (1 - \tau) f'(k)\big] \qquad \text{(C-locus)}$$

$$c = f(k) - (n + \delta)k \qquad \text{(K-locus)}$$

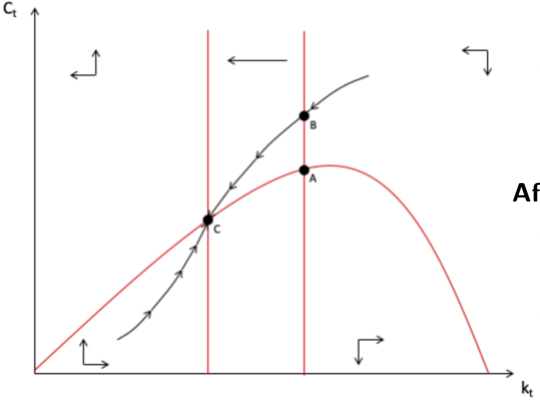
C-locus

- **Movement:** Shifts left.
- **Intuition:** The tax makes investing less attractive. The tax changes the relative prices - substitution effect.

K-locus

- **Movement:** Unchanged.
- **Intuition:** The tax revenue is rebated, why there is no income effect.

Problem 1c - Introducing capital income tax - Dynamics



On impact:

- **Capital:** Predetermined and thus Unchanged.
- **Consumption:** Substitution effect leads to increased consumption $\implies$ jump to B.

Afterwards:

- **Capital:** Capital deteriorates as investments are lower.
- **Consumption:** Consumption will decrease until the interest rate is high enough to ensure constant consumption.

Problem 1d - Savings rate

Show that the saving rate on the BGP, $s^* = \frac{y^* - c^*}{y^*}$ is decreasing in τ.

The steady state savings rate is defined as:

$$s^* = \frac{y^* - c^*}{y^*} = \frac{f(k^*) - \left(f(k^*) - (n+\delta)k^*\right)}{f(k^*)} = \frac{(n+\delta)k^*}{f(k^*)}$$

Taking the derivative of s^* wrt. τ yields:

$$\begin{aligned}
\frac{\partial s^*}{\partial \tau} = \frac{\partial\left(\frac{(n+\delta)k^*}{f(k^*)}\right)}{\partial \tau} &= \frac{(n+\delta)\frac{\partial k^*}{\partial \tau} f(k^*) - (n+\delta)k^* \frac{\partial f(k^*)}{\partial \tau}}{f(k^*)^2} \\
&= \frac{(n+\delta)\frac{\partial k^*}{\partial \tau} f(k^*) - (n+\delta)k^* \frac{\partial f(k^*)}{\partial k^*}\frac{\partial k^*}{\partial \tau}}{f(k^*)^2} \\
&= \frac{(n+\delta)}{f(k^*)^2}\frac{\partial k^*}{\partial \tau} \underbrace{\left(f(k^*) - k^* \frac{\partial f(k^*)}{\partial k^*}\right)}_{w^*}
\end{aligned}$$

$$\frac{\partial s^*}{\partial \tau} = \underbrace{\frac{(n+\delta)}{f(k^*)^2}}_{>0} \underbrace{\frac{\partial k^*}{\partial \tau}}_{<0 \text{ from 1b}} \underbrace{w^*}_{>0} < 0$$

Problem 1d - savings rate - derivations

The calculations are based on the quotient rule and the chain rule:

Quotient rule: $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Chain rule: $$\frac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big)g'(x)$$

We find the derivative of the numerator and the denominator:

$$\frac{\partial (n+\delta)k^*}{\partial \tau} = (n+\delta)\frac{\partial k^*}{\partial \tau} \qquad \text{and} \qquad \frac{\partial f(k^*)}{\partial \tau} = \underbrace{\frac{\partial f(k^*)}{\partial k^*}\frac{\partial k^*}{\partial \tau}}_{\text{chain rule}}$$

Using the quotient rule, we then find the derivative:

$$\frac{\partial\left(\frac{(n+\delta)k^*}{f(k^*)}\right)}{\partial \tau} = \frac{(n+\delta)\frac{\partial k^*}{\partial \tau}f(k^*) - (n+\delta)k^*\frac{\partial f(k^*)}{\partial k^*}\frac{\partial k^*}{\partial \tau}}{f(k^*)^2}$$

$$= \frac{(n+\delta)}{f(k^*)^2}\frac{\partial k^*}{\partial \tau}\underbrace{\left(f(k^*) - k^*\frac{\partial f(k^*)}{\partial k^*}\right)}_{w^*}$$

$$\frac{\partial s^*}{\partial \tau} = \underbrace{\frac{(n+\delta)}{f(k^*)^2}}_{>0}\underbrace{\frac{\partial k^*}{\partial \tau}}_{<0 \text{ from 1b}}\underbrace{w^*}_{>0} < 0$$

Problem 1d - Negative effect on savings rate

We have now shown that:

$$\frac{\partial s^*}{\partial \tau} < 0$$

Does this make intuitive sense?

The tax revenue is rebated and there is, therefore, no income effect. The only effect is a substitution effect, which leads s to be negatively affected by τ as τ is a tax on savings.

We know that the production function and capital depreciation is unchanged but the capital level is lower. Hence, it makes sense that the savings rate must be lower for the amount of savings to be lower.

Problem 1e - Taxes no longer rebated (1/2)

How do the answers to the previous questions change if the tax revenue is used to make gov. purchases that have no effect on household's utility?

C-locus

The first order conditions wrt. c_t and k_{t+1} are unchanged since T_t was taken as given. Hence, the Euler equation and C-locus are unchanged.

K-locus

The law of motion for capital changes:

$$\begin{aligned}
(1+n)k_{t+1} + c_t &= \left[1 - \delta + (1-\tau)f'(k_t)\right]k_t + w_t \\
c_t &= \left[1 - \delta - \tau f'(k_t)\right]k_t + \underbrace{f'(k_t)k_t + w_t}_{= r_t k_t + w_t = f(k_t)} - (1+n)k_{t+1} \\
c_t &= \left[1 - \delta - \tau f'(k_t)\right]k_t + f(k_t) - (1+n)k_{t+1}
\end{aligned}$$

For $k_{t+1} = k_t = k$ capital:

$$c = f(k) - \left[n + \delta + \tau f'(k)\right]k$$

The K-locus shifts downward.

Problem 1e - Taxes no longer rebated (2/2)

Then we find the derivative of the new steady state savings rate $\bar{s}$ wrt. τ:

$$\bar{s} = \frac{\left[n + \delta + \tau f'(\bar{k})\right] \bar{k}}{f(\bar{k})}$$

Differentiating wrt. τ yields:[2] ▸ More elaborate derivations

$$\frac{\partial \bar{s}}{\partial \tau} = \frac{(n+\delta)}{f(\bar{k})^2}\frac{\partial \bar{k}}{\partial \tau}\bar{w} + \frac{\left[f''(\bar{k})\frac{\partial \bar{k}}{\partial \tau}\tau\bar{k} + f'(\bar{k})\bar{k} + f'(\bar{k})\tau\frac{\partial \bar{k}}{\partial \tau}\right] f(\bar{k}) - f'(\bar{k})\frac{\partial \bar{k}}{\partial \tau} f'(\bar{k})\tau\bar{k}}{f(\bar{k})^2}$$

$$= \frac{(n+\delta)}{f(\bar{k})^2}\frac{\partial \bar{k}}{\partial \tau}\bar{w} + \frac{\left[f''(\bar{k})\frac{\partial \bar{k}}{\partial \tau}\tau\bar{k} + f'(\bar{k})\bar{k}\right] f(\bar{k}) + \overbrace{\left[f(\bar{k}) - f'(\bar{k})\bar{k}\right]}^{\bar{w}}\tau f'(\bar{k})\frac{\partial \bar{k}}{\partial \tau}}{f(\bar{k})^2}$$

$$= \underbrace{\frac{\left[n+\delta+\tau f'(\bar{k})\right]}{f(\bar{k})^2}\frac{\partial \bar{k}}{\partial \tau}\bar{w}}_{<0} + \underbrace{\frac{\left[\tau f''(\bar{k})\frac{\partial \bar{k}}{\partial \tau} + f'(\bar{k})\right] f(\bar{k})\bar{k}}{f(\bar{k})^2}}_{>0}$$

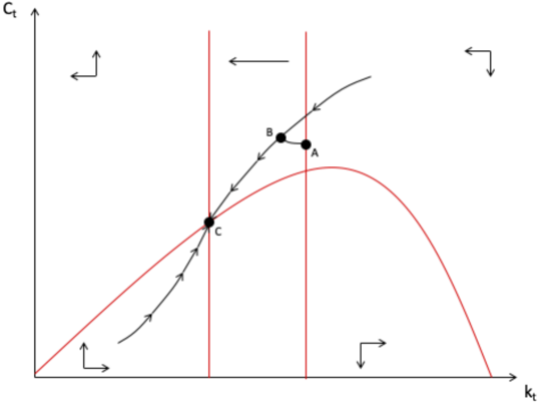
The effect of taxes on the savings rate is ambiguous. The positive effect is an income effect, which occurs because the taxes are no longer rebated. This leads the households to increase savings all else equal.

[2] Note: $f''(k) = \frac{\partial f'(k)}{\partial k} = \frac{\partial^2 f(k)}{\partial k^2}$

Problem 1f - Introducing $\tau > 0$ with early announcement

The households know that $\tau > 0$ will be introduced at time $t = T$.

- $t = T - 1$: The economy follows the old dynamics. Households increase consumption and move to point A.
- $t = T$: The tax is introduced and the C-locus shifts left. The economy follows the new dynamics The economy moves to point B.
- $t > T$: Convergence towards point C along the saddle path.



Problem 1f - Intuition elaboration

Before introduction $t = T - 1$:
It is announced that the tax will be introduced at time $t = T$. Households anticipate that the value of savings will decrease since the future interests will be taxed. Hence, they increase consumption. Capital is predetermined so they move along the C-locus to point A. The old dynamics are still in place so capital in period $t = T$ will decrease since they are above the K-locus.

At introduction $t = T$:
The tax is now introduced and the C-locus shifts left, so the new dynamics are in place. The economy moves to point B as the capital decrease than the last period. The economy is now on the saddle-path.

After introduction $t > T$:
The economy moves along the saddle-path as it converges to the new equilibrium at point C.

Problem 2

Firms face the following static problem:

$$\max_{L_t, K_t} \pi(K_t, L_t) = K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha} - w_t L_t - r_t K_t$$

Households face the following problem:

$$\max_{c_t, a_{t+1}} U = \sum_{t=0}^{\infty} \beta^t (1+n)^t \frac{c_t^{1-\sigma}}{1-\sigma}, \quad \sigma > 0, \quad \sigma \neq 1$$

$$\text{s.t. } (1+n)a_{t+1} = a_t R_t + w_t - c_t$$

Further, we know that:

$$a_t = k_t + b_{pt}$$
$$R_t = 1 + r_t - \delta$$

Problem 2a - FOCs for the firms

Find the first order conditions for the firms' maximization problem. Which must be the payment to households for the rent of their capital holdings?

Firms problem: Since we will get a different expression for A_t, we solve the problem without setting $A_t = 1$. Finding first order conditions of the profit function wrt. L_t and K_t yields:

$$\text{FOC wrt. } L_t: \quad w_t = A_t^{1-\alpha}(1-\alpha)K_t^{\alpha}L_t^{-\alpha} = A_t^{1-\alpha}(1-\alpha)k_t^{\alpha} \tag{6}$$
$$\text{FOC wrt. } K_t: \quad r_t = A_t^{1-\alpha}\alpha K_t^{\alpha-1}L_t^{1-\alpha} = A_t^{1-\alpha}\alpha k_t^{\alpha-1} \tag{7}$$

For $A_t = 1$, the FOCs become:

$$w_t = (1-\alpha)k_t^{\alpha}$$
$$r_t = \alpha k_t^{\alpha-1}$$

Hence, the households will be paid the marginal product of capital as rent for their capital holdings.

Problem 2b - No Ponzi game condition (NPGC)

Write the no Ponzi game condition. What is its economic meaning?

NPGC: The NPGC is given by:

$$\lim_{T\to\infty} q_T a_{T+1} \geq 0 \tag{NPGC}$$

Where $q_t = \prod_{i=1}^{t} \frac{(1+n)}{R_i}$.

The No Ponzi-Game Condition eliminates the households' ability to roll over debt indefinitely.

Problem 2c - Lagrange and FOCs for HH

Write the Lagrangian, find FOCs, Derive the Euler equation and interpret.

The Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t (1+n)^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \lambda_t \Big(a_t(1+r_t-\delta) + w_t - c_t - (1+n)a_{t+1} \Big) \right]$$

The first order conditions wrt. c_t and a_{t+1} are:

$$c_t^{-\sigma} = \lambda_t \tag{8}$$

$$\lambda_t = \beta \lambda_{t+1} \left[1 + r_{t+1} - \delta \right] \tag{9}$$

Combining the FOCs with r_t from problem 2a yields the Euler equation:

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \left[1 + \alpha k_{t+1}^{\alpha-1} - \delta \right] \tag{10}$$

The equation for consumption growth is then given by:

$$\frac{c_{t+1}}{c_t} = \left[\beta (1 + \alpha k_{t+1}^{\alpha-1} - \delta) \right]^{\frac{1}{\sigma}} \tag{11}$$

Interpretation is equivalent to PS2. σ measures risk-aversion and consumption smoothing preference. Low β leads to lower consumption growth, high r_t leads to higher consumption growth.

Problem 2d - Euler equation and capital accumulation

We note that in equilibrium there will be zero net borrowing $\implies b_{pt} = 0$ and thus $a_t = k_t$, which leads to the following capital accumulation:

$$
\begin{aligned}
(1+n)k_{t+1} &= (1 + r_t - \delta)k_t + w_t + -c_t \\
c_t &= (1-\delta)k_t + \underbrace{r_t k_t + w_t}_{=f(k_t)} -(1+n)k_{t+1} \\
c_t &= f(k_t) + (1-\delta)k_t - (1+n)k_{t+1} \\
c_t &= k_t^{\alpha} + (1-\delta)k_t - (1+n)k_{t+1}
\end{aligned}
$$

The Euler equation and capital accumulation determined the loci:

$$c = k^{\alpha} - (n+\delta)k \qquad \text{(k-locus)}$$

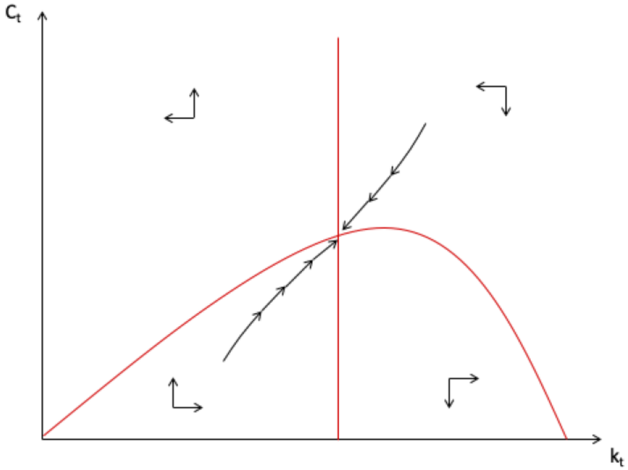
$$1 = \beta\left[1 - \delta + \alpha k^{\alpha-1}\right] \qquad \text{(c-locus)}$$

With steady state capital ▸ Derivation

$$k^* = \left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}}$$

The dynamics are almost identical to problem 1b ▸ C-dynamics ▸ K-dynamics

Problem 2d - Phase diagram



Note: The line with arrows is called the saddle path.

The economy will converge towards (k^*, c^*) along the saddle path. If the economy is not on the saddle path, the economy will diverge.

Problem 2e - Is it socially desirable

Is the equilibrium socially desirable? Justify your answer.

First welfare theorem: if markets are competitive and complete and there are no externalities (and if the number of agents is finite), then the decentralized equilibrium is Pareto-efficient - that is, it is impossible to make anyone better off without making someone else worse off.[3]

Since the conditions hold, the equilibrium is pareto-efficient and socially desirable.

[3] Lecture 2, slide 24.

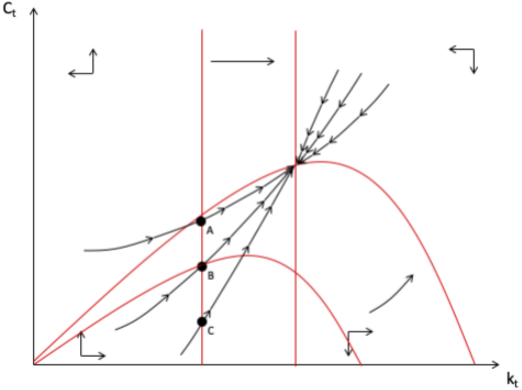
Problem 2f - Decrease in δ

Starting in SS there is an unexpected decrease in δ. Find the new SS graphically. What happens to consumption initially?

$\sigma < 1 \implies$ Substitution effect $>$ income effect $\implies$ Point C

$\sigma = 1 \implies$ Substitution effect $=$ income effect $\implies$ Point B

$\sigma > 1 \implies$ Substitution effect $<$ income effect $\implies$ Point A



Problem 2f - Effect on impact

The problem is identical to PS2, problem 1c but with a reversed sign.

- If $\theta < 1$: The substitution effect will dominate so household will decrease consumption on impact as a response to the lower depreciation rate of capital. The lower depreciation rate will make the effective return on savings higher, which makes future consumption more attractive. Hence, consumption will jump to point C.
- If $\theta = 1$: The substitution and income effect will cancel each other, which implies that the economy will jump to point B.
- If $\theta > 1$: The income effect will dominate, since households will dislike fluctuations in the consumption level. Households will be aware that the value of their savings has increased and they will, therefore, spend more to smooth their consumption. The economy will jump to point A.

The short-run effect depends on the functional form of the utility function of the household.

Problem 2g - Changing the technology function

$$A_t = \left(\frac{K_t^{AG}}{L_t^{AG}}\right)^{\gamma}$$

How does the phase diagram change when labour productivity is a function of aggregate capital? Does the social desirability change?

What is the intuition?

This is a standard way to introduce learning-by-doing in the economy. As aggregate capital increases, workers gets accustomed to working with capital and their productivity, therefore, increases.

Inserting the new A_t into equation 6 and 7 yields:

$$\text{FOC wrt. } L_t: \quad w_t = k_t^{\gamma(1-\alpha)}(1-\alpha)k_t^{\alpha} \quad = (1-\alpha)k_t^{\alpha+\gamma(1-\alpha)} \tag{12}$$

$$\text{FOC wrt. } K_t: \quad r_t = k_t^{\gamma(1-\alpha)}\alpha k_t^{\alpha-1} \quad = \alpha k_t^{(\alpha-1)(1-\gamma)} \tag{13}$$

Note: Firms do not account for the positive externality of capital per worker when maximizing profits but firms are identical so $\frac{K_t^{AG}}{L_t^{AG}} = k_t$.

Problem 2g

The new interest rate leads to the following C-locus:

$$1 = \beta(1 + \alpha k^{(\alpha-1)(1-\gamma)} - \delta)$$

With steady state capital:

$$k^* = \Big(\frac{\alpha\beta}{1-\beta(1-\delta)}\Big)^{\frac{1}{(1-\alpha)(1-\gamma)}} > \Big(\frac{\alpha\beta}{1-\beta(1-\delta)}\Big)^{\frac{1}{1-\alpha}}$$

Hence steady state capital increases.

Capital accumulation is now given by:

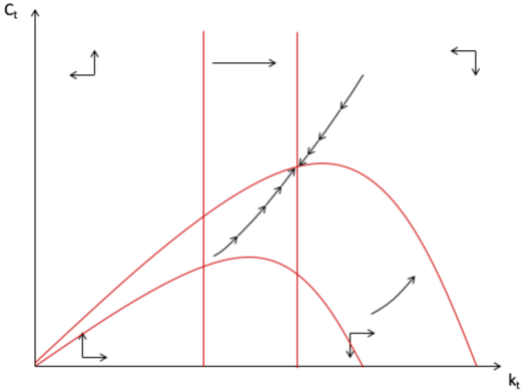
$$\begin{aligned} c_t &= (1-\delta+r_t)k_t + w_t - (1+n)k_{t+1} \\ &= (1-\delta)k_t + \alpha k_t^{(\alpha-1)(1-\gamma)} k_t + (1-\alpha)k_t^{\alpha+\gamma(1-\alpha)} - (1+n)k_{t+1} \\ &= (1-\delta)k_t + k_t^{\alpha+\gamma(1-\alpha)} - (1+n)k_{t+1} \end{aligned}$$

The K-locus is now given by:

$$c = k^{\alpha+\gamma(1-\alpha)} - (n+\delta)k$$

The exponent is greater than before, so the k-locus shifts upwards.

Problem 2g - Phase diagram and social desirability



Is the new equilibrium socially desirable? Why?

No, it is not longer socially desirable. The technology function introduces a positive externality, why **the first welfare theorem no longer holds**. The agents in the economy do not take into account the positive externality of capital per worker. This leads the capital per worker to be lower than what is socially desirable.

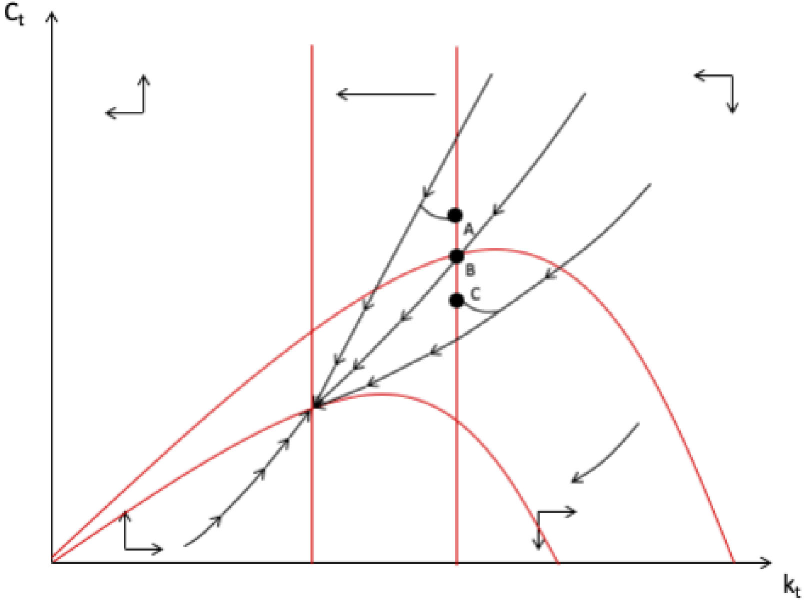
Problem 2h - Suddenly $\gamma = 0$

It is unexpectedly known that $\gamma = 0$ at time $t = T$ after being $\gamma > 0$ initially. What happens before and after T?

Agents are told at that at time T, the producitivity of the firms will change. Agents will react accordingly:

- $t < T$: Households adjust consumption when the information is received. Whether consumption will increase or decrease depends on the risk-aversion. The economy will momentarily diverge from old equilibrium based on the old dynamics.
- $t = T$: The loci move and the new saddle path occurs based on the new dynamics. The economy will immediately hit the new saddle path.
- $t > T$: The economy will converge towards the new equilibrium along the saddle path.

Problem 2h - Suddenly $\gamma = 0$ - Illustration



Derivation of steady state level capital

We isolate k in the c-locus:

$$
\begin{aligned}
1 &= \beta\left[1-\delta+\alpha k^{\alpha-1}\right] \\
\frac{1}{\beta} &= 1-\delta+\alpha k^{\alpha-1} \\
\alpha k^{\alpha-1} &= \frac{1}{\beta}-(1-\delta) \\
k^{\alpha-1} &= \frac{1-\beta(1-\delta)}{\alpha\beta} \\
k &= \left(\frac{1-\beta(1-\delta)}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} \\
k^{*} &= \left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

Back

Problem 1e - New savings rate - derivations (1/3)

For this one we use the chain rule, quotient rule and the product rule.[4]

$$\text{Product rule:} \qquad \frac{d}{dx}\Big(f(x)g(x)\Big) = f'(x)g(x) + f(x)g'(x)$$

The new savings rate can be divided into to fractions:

$$\frac{\left[n+\delta+\tau f'(\bar{k})\right]\bar{k}}{f(\bar{k})} = \frac{\left[n+\delta\right]\bar{k}}{f(\bar{k})} + \frac{\tau f'(\bar{k})\bar{k}}{f(\bar{k})}$$

We know the derivative of the first part from problem 1d. Using the product rule on the numerator of the second part yields:

$$\frac{\partial \tau f'(\bar{k})\bar{k}}{\partial \tau} = f'(\bar{k})\bar{k} + \tau \underbrace{\frac{\partial f'(\bar{k})}{\partial \bar{k}}\frac{\partial \bar{k}}{\partial \tau}}_{\text{Chain rule}}\bar{k} + \tau f'(\bar{k})\frac{\partial \bar{k}}{\partial \tau}$$

$$= f'(\bar{k})\bar{k} + \tau f''(\bar{k})\frac{\partial \bar{k}}{\partial \tau}\bar{k} + \tau f'(\bar{k})\frac{\partial \bar{k}}{\partial \tau}$$

We know from the chain rule that:

$$\frac{\partial f(\bar{k})}{\partial \tau} = \frac{\partial f(\bar{k})}{\partial \bar{k}}\frac{\partial \bar{k}}{\partial \tau} = f'(\bar{k})\frac{\partial \bar{k}}{\partial \tau}$$

[4] Chain rule and quotient rule are shown on slide 14.

Problem 1e - New savings rate - derivations (2/3)

We then proceed by finding the derivative of $\frac{\tau f'(\bar{k})\bar{k}}{f(\bar{k})}$ by using the quotient rule:

$$
\begin{aligned}
\frac{\partial\left(\frac{\tau f'(\bar{k})\bar{k}}{f(\bar{k})}\right)}{\partial \tau} &= \frac{\left[f'(\bar{k})\bar{k} + \tau f''(\bar{k})\frac{\partial \bar{k}}{\partial \tau}\bar{k} + \tau f'(\bar{k})\frac{\partial \bar{k}}{\partial \tau}\right] f(\bar{k}) - \tau f'(\bar{k})\bar{k} f'(\bar{k})\frac{\partial \bar{k}}{\partial \tau}}{f(\bar{k})^2} \\
&= \frac{\left[f'(\bar{k})\bar{k} + \tau f''(\bar{k})\frac{\partial \bar{k}}{\partial \tau}\bar{k}\right] f(\bar{k}) + \tau f(\bar{k}) f'(\bar{k})\frac{\partial \bar{k}}{\partial \tau} - \tau f'(\bar{k})\bar{k} f'(\bar{k})\frac{\partial \bar{k}}{\partial \tau}}{f(\bar{k})^2} \\
&= \frac{\left[f'(\bar{k}) + \tau f''(\bar{k})\frac{\partial \bar{k}}{\partial \tau}\right] f(\bar{k})\bar{k} + \overbrace{\left[f(\bar{k}) - f'(\bar{k})\bar{k}\right]}^{\bar{w}} \tau f'(\bar{k})\frac{\partial \bar{k}}{\partial \tau}}{f(\bar{k})^2} \\
&= \frac{\left[f'(\bar{k}) + \tau f''(\bar{k})\frac{\partial \bar{k}}{\partial \tau}\right] f(\bar{k})\bar{k}}{f(\bar{k})^2} + \bar{w}\frac{\tau f'(\bar{k})\frac{\partial \bar{k}}{\partial \tau}}{f(\bar{k})^2}
\end{aligned}
$$

Problem 1e - New savings rate - derivations (3/3)

Next we combine the derivatives:

$$\begin{aligned}
\frac{\partial \bar{s}}{\partial \tau} &= \frac{\partial\left(\frac{\tau f'(\bar{k})\bar{k}}{f(\bar{k})}\right)}{\partial \tau} \\
&= \frac{\partial\left(\frac{(n+\delta)\bar{k}}{f(\bar{k})}\right)}{\partial \tau} + \frac{\partial\left(\frac{\tau f'(\bar{k})\bar{k}}{f(\bar{k})}\right)}{\partial \tau} \\
&= \frac{(n+\delta)}{f(\bar{k})^2}\frac{\partial \bar{k}}{\partial \tau}\bar{w} + \bar{w}\frac{\tau f'(\bar{k})\frac{\partial \bar{k}}{\partial \tau}}{f(\bar{k})^2} + \frac{\left[f'(\bar{k})\bar{k} + \tau f''(\bar{k})\frac{\partial \bar{k}}{\partial \tau}\bar{k}\right]f(\bar{k})}{f(\bar{k})^2} \\
&= \underbrace{\frac{\left[n+\delta+\tau f'(\bar{k})\right]}{f(\bar{k})^2}\frac{\partial \bar{k}}{\partial \tau}\bar{w}}_{<0} + \underbrace{\frac{\left[\tau f''(\bar{k})\frac{\partial \bar{k}}{\partial \tau} + f'(\bar{k})\right]f(\bar{k})\bar{k}}{f(\bar{k})^2}}_{>0}
\end{aligned}$$

The effect of taxes on the savings rate is ambiguous. The positive effect is an income effect, which occurs because the taxes are no longer rebated. This leads the households to increase savings all else equal.

Back to 1e